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The p-moment Stability for Stochastic Hopfield-type Neural Networks with Impulses*

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Abstract: The paper mainly concerns the stochastically exponential stability in the p-moment of the equilibrium point for a class of stochastic Hopfield-type neural networks with impulses (SHNNswI), by constructing a Lyapunov functional and using the theory of stochastic analysis. The results is given in the form of inequalities, they can be verified much more easily. Furthermore, the algorithm is given, whose efficiency is illustrated by an example.

Keywords: Hopfield-type neural networks; Itô formula; Lyapunov functional; stochastic stability

Classification: AMS(2000) 60H15; 35R60 **CLC number:** O211.63; O175.13 **Document code:** A

1 Introduction

Recently, Hopfield-type neural networks have been studied extensively by many authors and found many application in different ares[1,2]. It is well known that studies on stability properties for neural dynamic system not only involve abrupt changes at certain moments due to instantaneous perturbations, which lead to impulsive effects, but also, involve environment disturbance, which lead to stochastic effects. Considering the factors above, the stability of the equilibrium point for SHNNswI are concerned by this paper.

The theory of impulsive differential equations represents a more natural framework for mathematical modeling of many real-world phenomena, such as population dynamic and the neural networks. In recent years, the impulsive differential equations have been extensively studied[3-5]. Furthermore, the dynamical behaviors of stochastic neural networks have emerged as a new subject of research in recent years. In particular, the stability criteria for stochastic neural networks becomes an attractive research problem of importance. Some initial results have just appeared, for example, in [6,7], for stochastic delayed Hopfield neural networks.

However, to the best of our knowledge, few scholars investigated the stability for stochastic Hopfield-type neural networks with impulses. In this paper, we mainly concerns on the stability for stochastic Hopfield-type neural network with impulses.

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2 Problem formulation and preliminaries

Consider the following stochastic Hopfield-type neural networks with impulses

$$\begin{cases} \mathrm{d}x_i(t) = \big[-c_i(t)x_i(t) + \sum\limits_{j=1}^{n} a_{ij}(t)f_j(x_j(t))\big]\mathrm{d}s \\ \qquad\qquad + \sigma(t, x_i(t))\mathrm{d}\omega_i(t), \quad t \neq t_k, \quad i = 1, 2, \cdots, n, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-), \quad k \in N, \\ x_i(t_0) = x_{i0}, \end{cases} \tag{1}$$

where $x_i(t)$ is the state of the neural network, $c_i(t) > 0$ is the neuron charging time continuous function, $a_{ij}(t)$ denotes the strength of the jth unit on the ith unit at time t. Impulsive moments $\Gamma = \{t_k; k \in N\}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots$; $\lim\limits_{k\to\infty} t_k = \infty$; $\Delta x_i(t_k)$ corresponds to the abrupt changes of the state at fixed impulsive moment t_k and $x_i(t_k) = x_i(t_k^+)$, $x_i(t_k^-)$ exists. Moreover, $\omega(t) = [\omega_1(t), \omega_2(t), \cdots, \omega_n(t)]^T$ is a Brownian motion defined on a complete space $(\Omega, \mathcal{F}; \mathcal{P})$ with a nature filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\omega(s) : 0 \leq s \leq t$.

Rewrite (1) in the vector form as follows

$$\begin{cases} \mathrm{d}x(t) = \big[-c(t)x(t) + A(t)f(x(t))\big]\mathrm{d}s + \sigma(t, x(t))\mathrm{d}\omega(t), \quad t \neq t_k, \quad i = 1, 2, \cdots, n, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-), \\ x(t_0) = x_0, \end{cases} \tag{2}$$

where

$$x(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)^T, \quad c(t) = \mathrm{diag}\big(c_1(t), c_2(t), \cdots, c_n(t)\big),$$

$$A(t) = \big(a_{ij}(t)\big)_{n\times n}, \quad \omega(t) = \big(\omega_1(t), \omega_2(t), \cdots, \omega_n(t)\big)^T.$$

In order to obtain the main results, the following assumptions and definitions are needed.

Assumption 1 The neuron activation functions all bounded and satisfy the following conditions

$$\big|f_j(x) - f_j(y)\big| \leq L_j^f |x - y|, \quad f_j(0) = 0, \quad x, y \in R_n, \quad j = 1, 2, \cdots, n, \tag{3}$$

where L_j^f is a constant.

Assumption 2 The mapping $\sigma : R^+ \times R_n \to R_n$ is globally Lipschitz continuous and satisfies the linear growth condition. Moreover

$$\mathrm{trace}\big[\sigma^T(t, x(t))\sigma(t; x(t))\big] \leq x^T(t)\Sigma x(t), \quad \sigma(t, 0) = 0, \tag{4}$$

where Σ is a nonnegative constant matrix with n dimension.

Assumption 3 $\Delta x_i(t_k) \leq \gamma_i x_i(t_k)$, $i = 1, 2, \cdots, n$, $k \in N$.

It is claimed that there exists a unique stochastic process $x(t)$ satisfying system (1) and all solutions of system (1) are continuous on the right and limitable on the left for Γ (see [8]). Obviously, $x(0) = 0$ is the equilibrium solution of system (1). In the rest of the paper, we will prove $x(0)$ is stable in p-moment.

Definition 1 There exist constants $\lambda > 0$ and $c > 0$ such that

$$E\big\|x(t,t_0,x_0)\big\|^p \leq c\|x_0\|^p \mathrm{e}^{-\lambda(t-t_0)}, \tag{5}$$

then $x(0)$ is stochastically exponential stability in p-moment.

3 Main results

The main results on exponential stability in p-moment of $x(0)$ for SHNNswI and the algorithm for illustrating the results are given in this section.

Theorem 1 Let Assumptions 1, 2, 3 be satisfied, if there exist symmetric positive definite matrix P and a constance $\lambda > 0$ such that

$$\lambda P - pPc(t) + pPA(t)L^f + \frac{p(p-1)}{2}\Sigma P < 0, \quad \big(1+\lambda_M(\gamma)\big)^p < \infty, \tag{6}$$

then the solution $x(0)$ of system (1) is stochastically exponential stability in p-moment.

Proof Consider the Lyapunov functional

$$V(t,x) = \mathrm{e}^{\lambda t} x(t)^{T\frac{p}{2}} P x(t)^{\frac{p}{2}}, \tag{7}$$

where P is a symmetric positive definite matrix and $\lambda > 0$ is a scale constance.

Computing the infinitesimal generator of $V(t,x)$ along $x(t,x(0))$

$$\begin{aligned} LV(t,x) &= \mathrm{e}^{\lambda t}\Big\{\lambda x(t)^{T\frac{p}{2}} P x(t)^{\frac{p}{2}} + p x(t)^{T(\frac{p}{2}-1)} P\big[-c(t)x(t) + A(t)f(x(t))\big]x(t)^{\frac{p}{2}} \\ &\quad + \frac{p(p-1)}{2}\mathrm{trace}\big(\sigma(t,x(t))^T x(t)^{T(\frac{p}{2}-1)} P x(t)^{(\frac{p}{2}-1)}\sigma(t,x(t))\big)\Big\} \\ &\leq \mathrm{e}^{\lambda t} x(t)^{T\frac{p}{2}} P\Big[\lambda - pc(t) + pA(t)L^f + \frac{p(p-1)}{2}\Sigma\Big]x(t)^{T\frac{p}{2}} \leq 0. \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} EV\big(t,x(t)\big) &= E\Big[V\big(t_0,x(0)\big) + \int_{t_0}^t LV(s,x)\mathrm{d}s\Big] \\ &\leq EV\big(t_0,x(0)\big) \leq \mathrm{e}^{\lambda t_0}\lambda_M(P)\|x(0)\|^P, \end{aligned} \tag{9}$$

and, by the definition of $V(t,x(t))$

$$E\|x(t,x_0)\|^p \leq \frac{\mathrm{e}^{-\lambda t}}{\lambda_m(P)} EV\big(t,x(t)\big) \leq \frac{\lambda_M(P)}{\lambda_m(P)}\mathrm{e}^{-\lambda(t-t_0)}\|x(0)\|^P, \tag{10}$$

where $\lambda_M(P)$, $\lambda_m(P)$ are maximum and minimum eigenvalue of P respectively.

Also, in view of Assumption 3, for all $t_k \in \Gamma$, one has

$$\begin{aligned} E\|x(t_k,x_0)\|^p &= E\sum_{i=1}^{n}\|x_i(t_k,x_0)\|^p = E\sum_{i=1}^{n}\|x_i(t_k^-,x_0)+\Delta x_i(t_k^-)\|^p \\ &\le E\sum_{i=1}^{n}(1+\gamma_i)^p\|x_i(t_k^-,x_0)\|^p \le \big(1+\lambda_M(\gamma)\big)^p E\|x(t_k^-,x_0)\|^p \\ &\le \big(1+\lambda_M(\gamma)\big)^p\frac{\lambda_M(P)}{\lambda_m(P)}\mathrm{e}^{-\lambda(t_k^- -t_0)}\|x(0)\|^P \\ &= \big(1+\lambda_M(\gamma)\big)^p\frac{\lambda_M(P)}{\lambda_m(P)}\mathrm{e}^{-\lambda(t_k-t_0)}\|x(0)\|^P, \end{aligned} \tag{11}$$

where $\gamma=\mathrm{diag}(\gamma_1,\gamma_2,\cdots,\gamma_n)$, $\lambda_M(\gamma)$ is the maximum eigenvalue of γ.

Thus, for all $t\in R^+$

$$E\|x(t,x_0)\|^p \le \frac{\mathrm{e}^{-\lambda t}}{\lambda_m(P)}EV\big(t,x(t)\big) \le c\,\mathrm{e}^{-\lambda(t-t_0)}\|x(0)\|^P, \tag{12}$$

where

$$c=\big(1+\lambda_M(\gamma)\big)^p\frac{\lambda_M(P)}{\lambda_m(P)}>0.$$

This completes the proof.

Remark 1 The result given by Theorem 1 contains the result of Theorem 1 in [9] as a special case, when $\sigma(x(t),x(t-\tau))=0$.

Remark 2 In this paper, the numerical simulation results are utilized firstly to prove the theory conclusion for stochastic neural network system, compared with [5,9]. Furthermore, the paper mainly concerns the impulse effects, which is not considered in [7].

The algorithm on the matrix inequality (6) is given by applying the continuous properties of matrix functions and the linear matrix inequality (LMI) technique.

Step 1 Let initial time $t_0=0$, $i=1$, maximum iterative number N_0.

Step 2 If there is one feasible solution P_0, λ_0 about the matrix inequality (6) by LMI toolbox in Matlab while take $t=t_0$, then go to Step 3, or the matrix inequality (6) don't hold and stop.

Step 3 Set $p=P_0$, $\lambda=\lambda_0$ in the matrix inequality (6), there must exists the number $\delta=\min\limits_{1\le k\le 2\times n}\{\delta_k\}$ where

$$\delta_k=\begin{cases}\min\{t>t_0 \mid D_k(t)=0\}, & \text{there exsit } t \text{ such that } D_k(t)=0,\\ +\infty, & \text{if } D_k(t)\ne 0.\end{cases}$$

Such that the determinant $D_k(t)$ of the kth leading principal minor of the matrix continuous function on left side of the matrix inequality (6) are negative in $[t_0,t_0+\delta)$. If $\delta=+\infty$, then the matrix inequality (6) hold and stop. If $\delta<+\infty$ and $i<N_0$, then let $t=t_0+\delta$, $i=i+1$ and go to Step 2. If $i\ge N_0$, then fail and stop.

To demonstrate the above algorithm and the result of Theorem 1, an example is given.

Example Assume that the network parameters of the neural networks (1) are chosen as

$$c(t) = \begin{bmatrix} 1+te^{-t} & 0 \\ 0 & 10 \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$p = 2, \quad \triangle x_1(t_k) = 0.2x_1(t_k^-), \quad \triangle x_2(t_k) = 0.3x_2(t_k^-), \quad t_k = 2k,$$

$$f_i(x_i) = \sin(x_i), \quad i = 1, 2.$$

Applying the above algorithm, we have:

Step 1 Let $t_0 = 0,\ i = 1,\ N_0 = 100$, go to the Step 2.

Step 2 Using LMI toolbox in the Matlab, we can get one feasible solution of (6)

$$P_0 = \begin{bmatrix} 1.2980 & 0.0131 \\ 0.0131 & 0.1136 \end{bmatrix}, \quad \lambda_0 = 1.0627,$$

go to the Step 3.

Step 3 Set $P = P_0,\ \lambda = \lambda_0$ in the inequality (6), we have

$$\lambda P - 2Pc(t) + 2PA(t)L^f + \Sigma P = \begin{bmatrix} -3.3170 & -0.2377 \\ -0.2377 & -2.3683 \end{bmatrix},$$

$$\left(1 + \lambda_M(\gamma)\right)^2 = (1 + 0.3)^2 = 1.6900.$$

It is obvious that all the determinant $D_k(t)$ are negative definite for all $t \in [0, +\infty)$, i.e., $\delta = \delta_k = +\infty$. Therefore, by Theorem 1, the origin solution $x(0)$ of the system (6) is stochastically exponential stability in the mean square. Figure 1 depicts time response of state variable x_1 and x_2 without and with impulse effects.

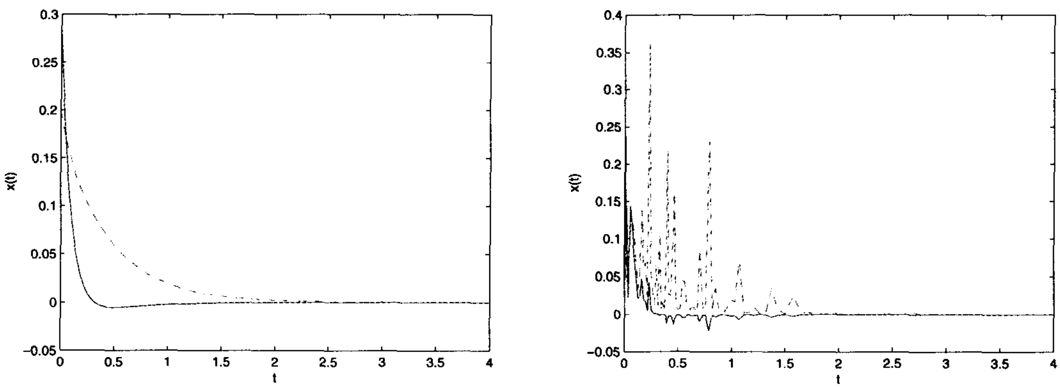

Figure 1: The trajectory of $x(t)$: left: without impulse effects; right: with impulse effects

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带脉冲的随机Hopfield型神经网络的p阶矩稳定性

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摘 要: 本文主要研究p阶矩意义下带脉冲的随机Hopfield型神经网络平衡点的随机指数稳定性。通过构造适当的Lyapunov泛函和利用随机分析理论，推导出平衡点稳定满足的条件。结果以不等式形式给出，使得结论更容易被验证。而且，本文给出求解算法，通过一数值算例，验证了算法的有效性。

关键词: Hopfield型神经网络；Itô公式；Lyapunov泛函；随机稳定性